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SIMPLIFIED SMALL SIGNAL GAIN CALCULATIONS IN FREE ELECTRON LASERS

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A method proposed recently is used for the evaluation of the small signal gain in various free electron laser configurations. The theory is applied to (a) the wiggler-free free electron laser with a uniform axial guide magnetic field and arbitrary direction of propagation of the amplified radiation and (b) the free electron laser with the axially modulated guide field (lowbitron). It is demonstrated that the new approach simplifies the gain calculation significantly in comparison with the traditional method.

Key words: correspondence principle, wiggler-free free electron laser, lowbitron.

Introduction

A novel method was recently proposed (1) to calculate small-signal gain in free electron lasers. The method combined classical and quantum mechanical formalisms by using the correspondence principle. This hybrid approach may be applied to the calculation of a gain in a variety of wave-generation configurations where the use of the

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conventional classical treatments usually leads to tedious algebraic manipulations.

In this paper the new method will be applied to the case of wave-amplification by guided relativistic electron beams. First the guide magnetic field will be assumed to be uniform and the wave will propagate at an arbitrary angle to the direction of the guide field. We will use the term "wiggler-free free electron laser" for this configuration (2). Useful examples of the parallel and nearly parallel propagation will be considered. The remaining part of the paper will deal with the small signal gain characterizing the free electron laser with the axially modulated guide field [lowbitron (3)].

The Method

In the previous paper (1), an expression was derived for the energy gain by an electron interacting with N uncorrelated plane electromagnetic waves in the presence of combined helical pump and axial guide magnetic field. Correct to the first order in terms of the amplitude of the perturbing electric field E_0, the gain was shown to obey

$$S = \sqrt{2\pi}\, A\eta \tag{1}$$

where

$$A = \frac{eE_0}{2\sqrt{2\pi}}\, B\, \frac{\sin\theta}{\theta} \tag{2}$$

and

$$\eta = \sum_{i=A}^{N} \cos\phi_i \tag{3}$$

Here ϕ_i are random quantities defined within an additive constant by the random relative phases of the waves. B in Eq. (2) is wL/u where w and u are the perpendicular and parallel electron velocities and L is the length of the laser. θ is $\nu\tau/2$ where $\nu = k_0 u - \omega(1 - u/c)$ is the resonance mismatch parameter (k_0 is the wavevector of the pump) and τ is the time the electron passes the interaction region. Using the correspondence principle and employing the detailed balancing arguments, the net energy gain was shown to be

$$\Gamma = -\frac{\omega_p^2 m}{8c}\frac{d}{d\varepsilon}\left(B^2 u F(\theta)\right) \tag{4}$$

where $\omega_p^2 = 4\pi ne^2/m$, n is the electron density, m and ε are the electron mass and energy respectively, and $F(\theta)$ is

$$F(\theta) = \left(\frac{\sin\theta}{\theta}\right)^2 \tag{5}$$

We will show in this paper that this calculation procedure can be applied to a variety of free electron laser configurations.

A Uniform Guide Magnetic Field

As a first example, let us consider an electron beam propagating along a uniform guide magnetic field $\underline{B}_0$

Let

$$\underline{B}_0 = \hat{e}_z B_0 \tag{6}$$

Assume that an electron interacts with N plane electromagnetic waves of equal amplitude E_0 and propagating along a given direction $\underline{k}/|\underline{k}|$.

$$\underline{E} = E_0 \sum_{j=1}^{N} \underline{P}_j \cos(\underline{k}\cdot\underline{r} - \omega t + \psi_j) \tag{7}$$

$\underline{P}_j$ and ψ_j are the random polarization vectors and phases respectively. In the zeroth order the electron velocity and position are described by

$$\begin{aligned} v_{\perp}(t) &= v_{\perp 0}\exp\left[i\phi + i\frac{\Omega}{\gamma_0}(t - t_0)\right] \\ r_{\perp}(t) &= -\frac{iv_{\perp 0}\gamma_0}{\Omega}\exp\left[i\phi + i\frac{\Omega}{\gamma_0}(t - t_0)\right] + r_{\perp 0} \end{aligned} \tag{8}$$

where complex notations are used and $v_{\perp} = v_x + iv_y$; $r_{\perp} = x + iy \cdot t_0$ in Eq. (8) is the time moment the electron enters the interaction region, $\Omega = eB_0/mc$ is the cyclotron frequency, γ_0 is the zeroth order value of $\gamma(= \varepsilon/mc^2)$

and $r_{\perp 0}$ is the coordinate of the gyration center of the electron. Let us assume now that the wave is a TE linearly polarized wave. Then the wave electric field is

$$E = iE_0 \sum_{j=1}^{N} \cos(k_z Z + k_\perp x - \omega t + \psi_j). \tag{9}$$

Consider now the electron energy conservation equation

$$\frac{d\varepsilon}{dt} = - e \, \mathrm{Re}(v_\perp E^*) \tag{10}$$

We will find the energy change of the electrons in the beam by integrating Eq. (10) along the unperturbed orbits described by $z = u(t - t_0)$ where u is the unperturbed axial velocity and x being the real part of $r_\perp$ in Eq. (8).

The interaction includes all the harmonics of the cyclotron frequency. Indeed,

$$v_\perp E^* = - \left(\frac{iv_{\perp 0}E_0}{2}\right) \sum_{j=1}^{N} \sum_{\ell=-\infty}^{\infty} J_{\ell-1}(\psi) \exp\{i[\ell\frac{\Omega}{\gamma_0} + (k_z u - \omega)]t' + i\alpha_{j\ell}\} \tag{11}$$

where t' is the time measured with respect to the moment the unperturbed ectron passes the center of the interaction region, $t' = t - t_0 - \tau/2$, $\tau = L/u$, L is the laser length, $\alpha_{j\ell} = [\ell\,\Omega/\gamma_0 + (ku - \omega)]\tau/2 + k_\perp x_c - \ell\pi/2 + \ell\phi + \pi/2 - \omega t_0 + \psi_j$, x_c = real $(r_{\perp 0})$, and $\psi = k_\perp v_{\perp 0}\gamma_0/2$.

The energy change of the electrons is

$$\Delta\varepsilon = -e \int_{-\tau/2}^{\tau/2} \mathrm{Re}(v_\perp E^*) dt' \tag{12}$$

If the nth harmonic is in resonance, namely

$$\nu_n = n\Omega/\gamma_0 + (k_z u - \omega) \cong 0 \tag{13}$$

the contribution of the other harmonics is negligible and (12) becomes

$$\Delta\varepsilon = -\frac{ev_{\perp 0}E_0}{2u} J_{n-1}(\psi)L \frac{\sin\theta}{\theta} \sum_{j=1}^{N} \sin\alpha_{j,n} \tag{14}$$

where $\theta = \nu_n \tau/2$. The last expression has the same form as Eq. (2) with $B = -(v_{\perp 0}/u)J_{n-1}(\psi)L$. The analysis of Ref. (1) thus may be applied here too. Following Eq. (4) the net power gain is

$$\Gamma = -\frac{\omega_p^2}{8c} J_{n-1}^2(\psi)L^2 \frac{d}{d\varepsilon}\left[\frac{v_{\perp 0}^2}{u} F(0)\right] \tag{15}$$

As in the cases considered in Ref. (1) the main contribution in the last expression comes from the derivative $d\theta/d\varepsilon$. The dependence of various electron parameters on its energy ε can be found by using the exact constant-of-motion characterizing an electron in combined uniform guide magnetic field and an electromagnetic wave with the electric vector perpendicular to both the guide field and the direction of propagation of the wave. In such a situation

$$\gamma(k_z c/\omega - u/c) = \text{const.} \tag{16}$$

Using the last expression,

$$\frac{du}{d\gamma} = \frac{(k_z c/\omega - u/c)}{\gamma} \tag{17}$$

and

$$\frac{d\theta}{d\varepsilon} = -\frac{L}{2mc^2} \frac{k_\perp^2}{\omega\gamma u} \tag{18}$$

where the small factor ν_n has been neglected. The gain, thus, becomes

$$\Gamma = \frac{\omega_p^2}{16}\left(\frac{v_{\perp 0}}{u}\right)^2 J_{n-1}(\psi) \frac{k_\perp^2}{\omega\gamma} \frac{dF}{d\theta} L^3 \tag{19}$$

This expression agrees with the previous result which was obtained for n=1, Ref. (2), derived for the case $k_\perp c/\omega \ll 1$, $\psi \cong 0$, and $J_{n-1}(\psi) \cong \delta_{n,1}$.

Another interesting example is when the wave propagates parallel to the magnetic field. Then $k_\perp = 0$ and Eq. (19) yields zero gain. Nevertheless, the gain proportional to L^2 still exists. As before, the dependences of $v_\perp$ and u on ε are found on using the constant-of-motion [Eq. (16)] where $k_z c/\omega = 1$, and $\gamma^2 = (1 - v_\perp^2 - u^2)^{-1/2}$. Eq. (15) for the gain then yields

$$\Gamma = -\frac{\omega_p^2}{8c^3}\frac{L^2}{u\gamma}\left\{\frac{\left[2(1-u) + v_{\perp 0}^2\left(\frac{2}{u^2} - \frac{1}{u} - 1\right)\right]\sin^2\theta}{\theta^2} - \frac{v_{\perp 0}^2\theta\sin 2\theta}{\theta^2 u^2}\right\} \tag{20}$$

which is similar to the result obtained by Ride and Colson (4).

The Lowbitron

A considerable study has been performed by M.I.T. researchers on the theory of the lowbitron (3). In this device the electrons are propagating along a modulated axial magnetic field which can be approximated (near the axis) by

$$\underline{B}_0 = \hat{e}_z B_0\left[1 + \frac{\delta B}{B_0}\sin k_0 z\right] \tag{21}$$

The momentum equation in such a magnetic field yields the following solution for the perpendicular velocity of the electrons

$$v_\perp = v_{\perp 0}\exp\left\{i\left[\phi + \frac{\Omega}{\gamma_0}(t-t_0) - \frac{\Omega}{\gamma_0}\frac{\delta B}{B_0}\frac{\cos k_0 u(t-t_0)}{k_0 u}\right]\right\} \tag{22}$$

Here we substituted $z = u(t - t_0)$ since the axial velocity of the electrons remains constant.

As before, the electron beam interacts with N plane electromagnetic waves with random polarizations propagating parallel to the magnetic field. Then the electric field vector that the electron experiences at z is

$$E = E_0 \sum_{j=1}^{N} e^{i\alpha_j}\cos(kz - \omega t + \psi_j') \tag{23}$$

Using the same notations as for the case of uniform guide magnetic field, we can express the work done on the electrons by the real part of $v_\perp E^*$, where

$$v_\perp E^* = \sum_{j=1}^{N} \left(\frac{v_{\perp 0} E_0}{2}\right) J_n(b) \exp[i(\xi_n t' + \beta_j)] \tag{24}$$

with $\xi_n = \omega(u/c - 1) + nk_0 u + \Omega/\gamma_0$; $\beta_j = n(k_0 u\tau/2 + \pi/2) + (\phi - \alpha_j + \Omega\tau/2\gamma_0) + (ku - \omega)\tau/2 - \omega\tau_0 + \psi_j'$; and $b = -\Omega\delta B/\gamma_0 B_0 k_0 u$. An assumption was made that $\xi_n \cong 0$, so that only the resonant terms in the sum in Eq. (24) were retained. The energy change of the electrons is found by integrating Eq. (12)

$$\Delta\varepsilon = -\frac{eE_0 v_{\perp 0}}{2u} J_n(b) L \frac{\sin\eta_n}{\eta_n} \sum_{j=1}^{N} \cos\beta_j \tag{25}$$

where $\eta_n = \xi_n \tau/2$. Again Friedland's analysis (1) is applied and yields for the gain

$$\Gamma = -\frac{\omega_p^2 m}{8c} \frac{(v_{\perp 0} L)^2}{u} J_n^2(b) \frac{dF}{d\eta_n} \frac{d\eta_n}{d\varepsilon} \tag{26}$$

Note that since the magnetic field has only an axial component, $\gamma(1 - u)$ is still a constant-of-motion. Using this fact, we obtain

$$\frac{d\eta_n}{d\varepsilon} \cong \frac{L}{2mc^2 \gamma u} nk_0 \tag{27}$$

and finally, the gain becomes

$$\Gamma = -\frac{\omega_p^2}{16c^3 \gamma} \left(\frac{v_{\perp 0}}{u}\right)^2 nk_0 J_n^2(b) \frac{dF}{d\eta_n} L^3 \tag{28}$$

The last expression for the gain agrees with that derived in Ref. (5).

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